A finite-step lattice walk in $\mathbb{Z}^3$ with no three collinear vertices

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Abstract

We give an explicit negative answer to Erdős Problem 193. We define a nested infinite discrete Hilbert path $H\colon \mathbb{N} \to \mathbb{N}^2$, select one terminal-steered index from every block of sixteen indices, and lift each selected point by its index. The resulting sequence in $\mathbb{Z}^3$ has successive steps in one fixed finite set and has no three collinear vertices. The obstruction to collinearity is a two-adic pair invariant: for indices with the same terminal orientation, a valuation of the planar chord equals the ordinary two-adic valuation of the index gap. Applying this identity to the two adjacent chords and their sum rules out a collinear triple. The complete construction has also been formalized in Lean 4 against Mathlib; the formal theorem uses no additional axioms.

1 Introduction

Let $S \subseteq \mathbb{Z}^3$ be finite. An infinite S-walk is a sequence $(P_a)_{a\geq 0}$ in $\mathbb{Z}^3$ such that $P_{a+1} - P_a \in S$ for every a. Erdős Problem 193 asks whether every such walk must contain three collinear points [1, 2]. Gerver and Ramsey proved the corresponding assertion in dimension two and proved a bounded-collinearity result in dimension three [2].

Our answer to the stated three-dimensional question is negative.

Theorem 1.1 (Main theorem)**.** *There are a finite set $S \subseteq \mathbb{Z}^3$ and an infinite sequence $P\colon \mathbb{N} \to \mathbb{Z}^3$ such that*

$$P_{a+1} - P_a \in S \qquad (a \in \mathbb{N}),$$

and no three distinct terms of P are collinear.

The construction uses the discrete Hilbert path, but the space-filling property is not the decisive ingredient. What matters is the finite-state recursion at the least significant base-4 digit. A terminal orientation state supplies a pairwise two-adic invariant. A two-digit steering rule then selects a syndetic subsequence on which that invariant applies to every pair. Lifting the planar path by its Hilbert index turns collinearity into a scaling relation between two planar chords; the pair invariant makes that relation impossible.

Section 2 fixes the infinite Hilbert indexing. Section 3 proves the pair law. Section 4 excludes triples, and Section 5 gives the bounded-gap selector and finite step menu. Section 6 records the formal and computational verification.

2 A nested discrete Hilbert path

2.1 The finite-state transducer

Write a square symmetry as a map on $\{0,1\}^2$. We use the eight symmetries

$$\begin{array}{c|c|c|c} I & (x,y) & X & (1-x,y) \\ Y & (x,1-y) & C & (1-x,1-y) \\ S & (y,x) & R & (1-y,x) \\ L & (y,1-x) & T & (1-y,1-x). \end{array}$$

Let

$$c_0 = (0,0), \quad c_1 = (0,1), \quad c_2 = (1,1), \quad c_3 = (1,0)$$

be the four child positions in traversal order, and set

$$r_0 = S, \qquad r_1 = I, \qquad r_2 = I, \qquad r_3 = T.$$

When the incoming state is g and the next digit is $q \in \{0,1,2,3\}$, the transducer emits $g(c_q)$ and changes state to $g \circ r_q$. The complete table, with entries "emitted bits / next state", is

state	0	1	2	3
I	$00/S$	$01/I$	$11/I$	$10/T$
X	$10/R$	$11/X$	$01/X$	$00/L$
Y	$01/L$	$00/Y$	$10/Y$	$11/R$
C	$11/T$	$10/C$	$00/C$	$01/S$
S	$00/I$	$10/S$	$11/S$	$01/C$
R	$10/X$	$00/R$	$01/R$	$11/Y$
L	$01/Y$	$11/L$	$10/L$	$00/X$
T	$11/C$	$01/T$	$00/T$	$10/I$

For a length-ℓ base-4 word $w = q_{\ell-1} \cdots q_1 q_0$, read from most significant to least significant, starting in state I. If the emitted pairs, indexed by their binary place, are (x_j, y_j) for $0 \le j < \ell$, define

$$H_\ell(w) = \left(\sum_{j=0}^{\ell-1} x_j 2^j, \sum_{j=0}^{\ell-1} y_j 2^j \right). \tag{1}$$

Thus order one traverses $(0,0), (0,1), (1,1), (1,0)$. The recursion above is the standard discrete Hilbert recursion [3, 7].

Lemma 2.1 (Finite-order Hilbert properties)**.** *For every $\ell \ge 0$, the map from length-ℓ base-4 words to $\{0, \ldots, 2^\ell - 1\}^2$ given by* (1) *is a bijection. Words representing consecutive integers map to lattice neighbors. Furthermore, prepending two zero digits does not change the represented point.*

Proof. The first two assertions follow by induction on ℓ. At the inductive step the four first digits place rotated or reflected copies of the order-ℓ path in the four quadrants. The endpoint identities for $r_0 = S, r_1 = r_2 = I, r_3 = T$ show that consecutive copies meet along one lattice edge. These are exactly the four cases in the displayed transition table. For the last assertion, the first zero emits 00 and changes I to S; the second emits 00 and changes S back to I. The remaining word is therefore evaluated in its original state and at its original binary places. □

For $n \in \mathbb{N}$, write n in base 4 and prepend zero digits to any sufficiently large even length. Lemma 2.1 makes the following definition independent of that length.

Definition 2.2. The nested discrete Hilbert path $H\colon \mathbb{N} \to \mathbb{N}^2$ is the finite-order point of the even-padded base-4 word for n.

The map H is injective, and

$$\|H(n+1) - H(n)\|_1 = 1 \qquad (n \in \mathbb{N}). \tag{2}$$

2.2 Terminal states

Only digits 0 and 3 alter the orientation state. The two transformations S and T are commuting involutions and $ST = C$. Hence the reachable terminal states form the Klein four-group

$$K = \{I, S, T, C\} \cong \mathbb{F}_2^2.$$

For a finite word w, let $N_0(w)$ and $N_3(w)$ count its zero and three digits. Its terminal state is

$$\sigma(w) = S^{N_0(w) \bmod 2} T^{N_3(w) \bmod 2}. \tag{3}$$

Two leading zero digits preserve this state, so even padding defines $\sigma(n) \in K$ for every $n \in \mathbb{N}$.

For later use, observe that every fixed-digit transition $g \mapsto g \circ r_q$ is invertible. More strongly, knowing the outgoing state h and digit q recovers the incoming state as $h \circ r_q$, since every r_q is an involution. The emitted bit pair can then be recovered as

$$(h \circ r_q)(c_q) = h(c_q), \tag{4}$$

because r_q fixes its child corner c_q. This backward-emission identity is the mechanism behind the pair law.

3 The two-adic pair law

Put $\nu_2(0) = \infty$. For a nonzero vector $u = (u_x, u_y) \in \mathbb{Z}^2$, define

$$p(u) = \min\{\nu_2(u_x), \nu_2(u_y)\}, \tag{5}$$

$$\epsilon(u) = \begin{cases} 1, & \nu_2(u_x) = \nu_2(u_y), \\ 0, & \nu_2(u_x) \neq \nu_2(u_y), \end{cases} \tag{6}$$

$$V(u) = 2p(u) + \epsilon(u). \tag{7}$$

Lemma 3.1 (Scaling)**.** *For $u \neq 0$ and every positive integer k,*

$$V(ku) = V(u) + 2\nu_2(k).$$

Proof. Multiplication by k adds $\nu_2(k)$ to each finite coordinate valuation. It therefore adds $\nu_2(k)$ to $p(u)$ and leaves the equality bit $\epsilon(u)$ unchanged. □

Lemma 3.2 (First mismatch)**.** *Let two even-padded base-4 words have the same terminal state, and let j be their first differing digit position counted from the low end. Their Hilbert coordinates agree below binary place j. At place j, exactly one coordinate bit differs when the two base-4 digits have odd difference, and both coordinate bits differ when their difference is 2 modulo 4.*

Proof. The words agree below position j. Starting from their equal terminal states, invert the common low suffix. The outgoing state immediately after position j is therefore one common state h. Formula (4) says that the two bits emitted at the mismatch are $h(c_d)$ and $h(c_e)$, where $d \neq e$ are the two digits. The four child corners form a cyclic Gray code: c_d and c_e differ in exactly one coordinate when $d - e$ is odd, and in both coordinates when $d - e \equiv 2 \pmod 4$. Every square symmetry preserves these two alternatives. The common lower suffix begins in the same state and reads the same digits, so all lower emitted bits agree. □

Theorem 3.3 (Same-terminal-state pair law)**.** *If $m \neq n$ and $\sigma(m) = \sigma(n)$, then*

$$V\big(H(n) - H(m)\big) = \nu_2(n - m), \tag{8}$$

where the right side means $\nu_2(|n - m|)$ if $m > n$.

Proof. By symmetry assume $m < n$. Pad both base-4 words to one common even length, and let j be their first differing position from the low end. Write their digits there as a_j and b_j. Then

$$n - m = 4^j K, \qquad K \equiv b_j - a_j \pmod 4.$$

If $b_j - a_j$ is odd, then $\nu_2(n - m) = 2j$. By Lemma 3.2, exactly one coordinate of $H(n) - H(m)$ has its first nonzero binary digit at position j; the other has larger valuation. Higher binary places contribute multiples of 2^{j+1} and cannot alter this fact. Thus $p = j$, $\epsilon = 0$, and $V = 2j$.

If $b_j - a_j \equiv 2 \pmod 4$, then $\nu_2(n - m) = 2j + 1$. Both coordinate differences have their first nonzero binary digit at position j, so $p = j$, $\epsilon = 1$, and $V = 2j + 1$. These cases exhaust unequal base-4 digits. □

4 Excluding collinear triples

Theorem 4.1 (A terminal class is triple-free)**.** *Fix $s \in K$. The lifted set*

$$\mathcal{L}_s = \{(H(n), n) \in \mathbb{Z}^3 : \sigma(n) = s\}$$

contains no three collinear points.

Proof. Suppose for contradiction that $a < b < c$ have terminal state s and that their lifts are collinear. Set

$$A = b - a, \quad B = c - b, \quad X = H(b) - H(a), \quad Y = H(c) - H(b).$$

Equality of the lifted chord slopes gives

$$BX = AY. \tag{9}$$

Write $A = gr$, $B = gs$, where $g = \gcd(A, B)$ and $\gcd(r, s) = 1$. Equation (9) and coprimality imply that there is a nonzero $W \in \mathbb{Z}^2$ with

$$X = rW, \qquad Y = sW.$$

Let $\gamma = \nu_2(g)$, $\alpha = \nu_2(r)$, and $\beta = \nu_2(s)$. Theorem 3.3 and Lemma 3.1, first for (a, b) and then for (b, c), give

$$\gamma + \alpha = V(X) = V(W) + 2\alpha,$$
$$\gamma + \beta = V(Y) = V(W) + 2\beta.$$

Hence $\alpha = \beta$. Since r and s are coprime, both valuations are zero, and consequently $V(W) = \gamma$.

Now use the endpoint pair (a, c). Its planar chord is $X + Y = (r + s)W$ and its index gap is $g(r + s)$. With $\delta = \nu_2(r + s)$, the pair law and scaling law yield

$$\gamma + \delta = V((r + s)W) = V(W) + 2\delta = \gamma + 2\delta.$$

Thus $\delta = 0$. But r and s are both odd, so $r + s$ is even and $\delta \geq 1$, a contradiction. □

The use of all three pair laws—for gaps A, B, and $A + B$—is essential. A pairwise direction restriction alone would not rule out every triple.

5 A bounded-gap selector and the walk

For a block index $a \geq 0$, choose a correction $\rho(\sigma(a))$ by

q	I	S	T	C
$\rho(q)$	$5 = 11_4$	$1 = 01_4$	$13 = 31_4$	$3 = 03_4$

and define

$$n_a = 16a + \rho(\sigma(a)). \tag{10}$$

The two low base-4 digits of n_a have terminal contribution $\sigma(a)^{-1} = \sigma(a)$, so

$$\sigma(n_a) = I. \tag{11}$$

Moreover, ρ takes values in $\{1, 3, 5, 13\}$, and therefore

$$4 \leq n_{a+1} - n_a \leq 28. \tag{12}$$

Define

$$P_a = (H(n_a), n_a) \in \mathbb{Z}^3. \tag{13}$$

By Theorem 4.1, no three distinct P_a are collinear. By the unit-step property (2) and the triangle inequality,

$$\|H(n_{a+1}) - H(n_a)\|_1 \leq n_{a+1} - n_a \leq 28.$$

Thus every successive displacement belongs to the fixed finite set

$$S = \{(d_x, d_y, d_z) \in \mathbb{Z}^3 : |d_x| + |d_y| \leq 28,\ 4 \leq d_z \leq 28\}. \tag{14}$$

The z-coordinate is strictly increasing, so the sequence is infinite and has no repeated vertices. Equations (13) and (14) prove Theorem 1.1.

Remark 5.1. The displayed set S is a convenient finite superset of the realized step vectors, not a minimal menu. The independently reconstructed first 500,000 steps realize 16 distinct vectors.

6 Formalization and reproducibility

The repository [6] contains a Lean 4 formalization in `formal/Hilbert193`. It uses Lean 4.33.0 and the corresponding pinned Mathlib revision. The final theorem is `Hilbert193.erdos193_unconditional` in `Hilbert193/Continuity.lean`. It constructs:

1. a finite set of integer three-dimensional step vectors;
2. an explicit sequence $\mathbb{N} \to \mathbb{Z}^3$ whose successive displacements lie in that set; and
3. a proof that every ordered triple of distinct indices fails the exact integer collinearity equations.

The axiom audit in `Hilbert193/AxiomAudit.lean` reports only `propext`, `Classical.choice`, and `Quot.sound`, the standard axioms used by Mathlib. No theorem in the construction depends on a finite computation.

From the repository root, the formal proof is checked by

```
cd formal/Hilbert193
lake build
lake env lean Hilbert193/AxiomAudit.lean
```

The independent finite reconstruction is checked by

```
python3 verify_hilbert_construction.py hilbert-193-500k.jsonl \
  --steps 500000 --fresh
```

It reconstructs 500,001 vertices and checks the defining selector, coordinates, and fixed-menu condition for the recorded finite artifact. It does not enumerate triples. A separate exact direction-hash scan is run by

```
clang++ -O3 -std=c++17 -pthread verify_hilbert_exhaustive.cpp \
  -o verify_hilbert_exhaustive
./verify_hilbert_exhaustive hilbert-193-500k.jsonl
```

For each vertex, this scan reduces the vectors to all earlier vertices to canonical primitive integer directions. A repeated direction is equivalent to a collinear triple. The completed four-worker run checked all 125,000,250,000 pairs in 6299.67 seconds and found no repetition. The artifact SHA-256 is

`6f8fdf59b7ecef5533b8a65265d80df9`
`61f67a955dabaf9d5bb973d7ae143d63`.

Concatenating the two lines gives the digest. These computations are implementation checks and finite evidence; the infinite theorem rests on the arguments above and their Lean formalization.

The Lean predicate packages collinearity as the two integer cross-multiplication equalities forced by the strictly increasing third coordinate. This is the constructive, denominator-free form of ordinary affine collinearity in $\mathbb{Q}^3$ for the ordered points in this walk.

Acknowledgments and disclosure

The working stack included Paseo, Oh My Pi, and OpenAI GPT-5.6 for proof exploration, Lean code drafting, computational verification tooling, visualization, and manuscript preparation. The author directed the work, checked the generated artifacts, and takes responsibility for every claim. The proof is kernel-checked, but at the date of this preprint it has not yet completed external mathematical peer review.

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